SURVEY OF GRID POINTS IN A SINGLE-DIGIT DOMAIN GENERATING SPECIAL ANGLES

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When asking for the angle between vectors, it is beneficial for students if the answer is simply a specific "special angle," such as $\pi/6$, $\pi/4$, $\pi/3$. The question becomes even better if the vectors are expressed using small integers. Based on this principle, a theoretical survey for grid points in 2D space and a numerical survey in 3D space were conducted, focusing especially on the "single-digit domain."

The theoretical exploration in 2D was successfully completed and yielded a general conclusion: there exists a two-parameter family of integer vectors forming an angle of $\pi/4$, while no integer-coordinate vectors form $\pi/6$ or $\pi/3$. Exactly 10 distinct pairs realizing $\pi/4$ were identified, in the single-digit domain. The numerical exploration in 3D detected 28 pairs of vectors within the single-digit domain. A total of 38 "primary minimal pairs" were found. To complement and extend the primary minimal pairs, two theoretical tools were introduced: "complementary integer adjustment" (CIA) and "rational orthogonal transformation" (ROT). The outcomes obtained via CIA and ROT lie outside the single-digit domain with one overlapped. As a byproduct of CIA and ROT, additional 48 pairs of vectors just outside the single-digit domain were also found, forming special angles.

The results of this study provide a useful resource for educators designing class materials and exam questions related to vector calculations. They enable the creation of exercises where students can focus on fundamental vector operations without computational distractions. Furthermore, the theoretical approach in the 2D case suggests the possibility of developing advanced questions that ask students to prove the nonexistence of integer vector pairs forming $\pi/12$, $\pi/8$, $\pi/5$, and related angles.

Extending this analysis to 3D introduces additional mathematical challenges, as the problem reduces to solving a fourth-degree algebraic equation with six unknown integers. While a complete theoretical classification remains open, this work suggests possible deeper connections to number theory.

Future work includes implementing the proposed question design in classroom settings, especially in linear algebra courses, to evaluate its impact on students' understanding.

Keywords: *linear algebra, dot product, angle between integer vectors*

Motivation

The author teaches mathematics at a technological college in Japan, covering subjects such as linear algebra, calculus, and engineering mathematics. He frequently designs quizzes and problems for his students, adhering to the principle:

Principle: *Simple questions with simple answers encourage students to learn more.*

He believes that well-structured problems should have clear and straightforward solutions, avoiding unnecessary complexity. This approach allows students to focus on fundamental mathematical concepts without being hindered by excessive computational difficulties.

A typical example involving vectors in linear algebra is:

Q1: *Find the angle between two vectors* $\overrightarrow{\mathrm{AB}}$ *and* $\overrightarrow{\mathrm{AC}}$ *for given points* A, B, *and* C.

If the points are chosen arbitrarily, the answer might involve inverse trigonometric functions, such as

$$\cos^{-1}\frac{2}{3},$$

which can be cumbersome for students to evaluate. Instead, angles corresponding to

$$\cos^{-1}\frac{\sqrt{3}}{2}, \qquad \cos^{-1}\frac{\sqrt{2}}{2}, \qquad \cos^{-1}\frac{1}{2}$$

are more intuitive, as they correspond to $\pi/6$, $\pi/4$, $\pi/3$, respectively. The author refers to these as "special angles."

Q1 is a simple yet effective problem, particularly when the answer is a special angle. Moreover, the question becomes more accessible if A, B, and C are located at grid points in a coordinate system. In line with the **Principle**, the author prefers using single-digit numbers in questions. In this article, the term "single-digit domain" refers to the region defined by integer pairs (m, n) in 2D space or triples (l, m, n) in 3D space, where $0 \le l, m, n \le 9$ and $l, m,$ and n are mutually prime. We consider $\boldsymbol{u} = (m, n)$ or $\boldsymbol{u} = (l, m, n)$ as an input within this domain, and aim to find a corresponding $\boldsymbol{v}$ within the same region. The same approach is applied to the 3D case.

Motivated by this, the author conducted a survey of grid points in 2D and 3D spaces that generate special angles. Initially, he consulted an AI chatbot to explore potential grid points. He then used the chatbot to assist in writing a Python script for systematic exploration. Finally, he attempted a theoretical approach to solving the problem himself. While the 2D case proved relatively straightforward, the 3D case involved significant mathematical challenges. Since deriving a general but highly complex solution would not be suitable for educational purposes, a numerical survey of grid points was considered the appropriate approach. This paper presents the numerical findings for the 3D case, along with a discussion of potential theoretical methods.

Some Notations and Preparation

Suppose three points A, B, and C with assuming

$$\overrightarrow{\mathrm{AB}} = \boldsymbol{u} \qquad \text{and} \qquad \overrightarrow{\mathrm{AC}} = \boldsymbol{v}.$$

We introduce a notation $A(\boldsymbol{u},\boldsymbol{v})$ to indicate the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, especially for the 2D case $\boldsymbol{u} = (u_1, u_2)^T$ and $\boldsymbol{v} = (v_1, v_2)^T$:

$$A(\boldsymbol{u},\boldsymbol{v}) = A\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

and for the 3D case $\boldsymbol{u} = (u_1, u_2, u_3)^T$ and $\boldsymbol{v} = (v_1, v_2, v_3)^T$:

$$A(\boldsymbol{u},\boldsymbol{v}) = A\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}.$$

We regard $\boldsymbol{u}$ as an input with letting $u_1 = m$ and $u_2 = n$, and $\boldsymbol{v}$ as an output to satisfy a relation:

$$A(\boldsymbol{u},\boldsymbol{v}) = \theta$$

where $\theta = \angle\mathrm{BAC}$ will be a special angle.

Swapping the rows does not change the angle:

$$A\begin{pmatrix} u_2 & v_2 \\ u_1 & v_1 \end{pmatrix} = A\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}, \tag{1}$$

or

$$A\begin{pmatrix} u_2 & v_2 \\ u_3 & v_3 \\ u_1 & v_1 \end{pmatrix} = A\begin{pmatrix} u_3 & v_3 \\ u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \cdots = A\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}, \tag{2}$$

Thus, we can skip checking the triangular half of grid points using this symmetry with respect to a line $y = x$. Hence, we redefine the "2D single-digit domain" as

$$D_2 \stackrel{\text{def}}{=} \{(m,n)\epsilon\mathbb{Z}^2 \mid 0 \le m \le n \le 9\}$$

and "3D single-digit domain" as

$$D_3 \stackrel{\text{def}}{=} \{(l,m,n)\epsilon\mathbb{Z}^3 \mid 0 \le l \le m \le n \le 9\}$$

The following properties are trivial if $a, b > 0$:

$$A(a\boldsymbol{u},\boldsymbol{v}) = A(\boldsymbol{u},b\boldsymbol{v}) = A(\boldsymbol{u},\boldsymbol{v}) \tag{3}$$

for which we assume that coordinates of $\boldsymbol{u}$ or $\boldsymbol{v}$ should be mutually prime, and

$$A(\boldsymbol{v},\boldsymbol{u}) = A(\boldsymbol{u},\boldsymbol{v}) \tag{4}$$

for which we can skip $\boldsymbol{u}$ if it is equivalent with a previously found $\boldsymbol{v}$.

Theoretical Results in D_2

Suppose $\overrightarrow{\mathrm{AB}} = \boldsymbol{u} = (m,n)$ and $\overrightarrow{\mathrm{AC}} = \boldsymbol{v} = (v_1, v_2)$ in D_2 with $\theta = \angle\mathrm{BAC} \in [0,\pi]$. By combining two formulae

$$\boldsymbol{u}\cdot\boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$$

and

$$\|\boldsymbol{u}\times\boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta,$$

we find

$$\tan\theta = \frac{\|\boldsymbol{u}\times\boldsymbol{v}\|}{\boldsymbol{u}\cdot\boldsymbol{v}} = \frac{|mv_2 - nv_1|}{mv_1 + nv_2}$$

Let $t = \tan\theta$, then find $\pm t(mv_1 + nv_2) = mv_2 - nv_1$, which leads to

$$\frac{v_2}{v_1} = \frac{n \pm tm}{m \mp tn} \tag{5}$$

Proposition: *No grid points in a 2D space form an angle $\pi/3$ and $\pi/6$.*

proof: The angle $\theta = \pi/3$, i.e., $t = \sqrt{3}$ derives

$$\frac{v_2}{v_1} = \frac{n \pm \sqrt{3}\,m}{m \mp \sqrt{3}\,n} = \frac{4mn \pm \sqrt{3}(m^2+n^2)}{m^2 - 3n^2}$$

which has no integer solutions of v_1 and v_1 because $m^2 + n^2 \ne 0$. Thus, we conclude that no grid point triples form an angle $\theta = \pi/3$.
Similarly, no triples of grid points to form $\theta = \pi/6$ because $t = 1/\sqrt{3}$ in (5) leads to

$$\frac{v_2}{v_1} = \frac{\sqrt{3}\,n \pm m}{\sqrt{3}\,m \mp n} = \frac{4mn \pm \sqrt{3}(m^2+n^2)}{3m^2 - n^2}$$

■

Contrarily, the angle $\theta = \pi/4$ can be formed by 10 pairs of $(\boldsymbol{u},\boldsymbol{v})$ in D_2. In the case, we find $t = 1$, leading to

$$\frac{v_2}{v_1} = \frac{n \pm m}{m \mp n} = \frac{m+n}{m-n} \text{ or } \frac{n-m}{m+n}$$

Under the assumption $m \le n$ in D_2, we adopt the second form in our survey:

$$\frac{v_2}{v_1} = \frac{n-m}{m+n} \tag{6}$$

Furthermore, the ratio $v_1{:}v_2$ should also be in its simplest form, just like $m{:}n$.

We begin the survey in ascending order of m and n within the upper triangular region of **Table 1** excluding $m = n = 0$. The process follows these steps:

1. Skip a grid point (m', n') if m'/n' is equal to a previously obtained value of m/n
   (yellow cells in **Table 1)**
2. Skip a grid point (m', n') if $\begin{pmatrix} m' & m'+n' \\ n' & n'-m' \end{pmatrix}$ is equivalent to a previously obtained matrix after swapping rows or columns.
   (green cells in **Table 1**)
3. Skip a grid point (m', n') if $m' > 9$ or $n' > 9$ after division by GCD(m', n').
   (orange cells in **Table 1**)

Table 1: Checking process of grid points in D_2

m \ n	1	2	3	4	5	6	7	8	9
0	$\begin{pmatrix}0&1\\1&1\end{pmatrix}$	~(0,1)	~(0,1)	~(0,1)	~(0,1)	~(0,1)	~(0,1)	~(0,1)	~(0,1)
1	~(0,1)	$\begin{pmatrix}1&2\\3&1\end{pmatrix}$	~(1,2)	$\begin{pmatrix}1&4\\5&3\end{pmatrix}$	$\begin{pmatrix}1&5\\3&2\end{pmatrix}$	$\begin{pmatrix}1&6\\7&5\end{pmatrix}$	$\begin{pmatrix}1&7\\4&3\end{pmatrix}$	$\begin{pmatrix}1&8\\9&7\end{pmatrix}$	$\begin{pmatrix}1&9\\5&4\end{pmatrix}$
2		~(1,1)	~(1,5)	~(1,2)	$\begin{pmatrix}2&5\\7&3\end{pmatrix}$	~(1,3)	$\begin{pmatrix}2&7\\9&5\end{pmatrix}$	~(1,4)	$\begin{pmatrix}2&9\\11&7\end{pmatrix}$
3			~(1,1)	~(1,7)	~(1,4)	~(1,2)	~(2,5)	$\begin{pmatrix}3&8\\11&5\end{pmatrix}$	~(1,3)
4				~(1,1)	~(1,9)	~(2,3)	$\begin{pmatrix}4&7\\11&3\end{pmatrix}$	~(1,2)	$\begin{pmatrix}9&4\\13&5\end{pmatrix}$
5					~(1,1)	$\begin{pmatrix}5&6\\11&1\end{pmatrix}$	~(1,6)	$\begin{pmatrix}8&5\\13&3\end{pmatrix}$	~(2,7)
6						~(1,1)	$\begin{pmatrix}7&6\\13&1\end{pmatrix}$	~(3,4)	~(2,3)
7							~(1,1)	$\begin{pmatrix}7&8\\15&1\end{pmatrix}$	~(1,8)
8								~(1,1)	$\begin{pmatrix}8&9\\17&1\end{pmatrix}$
9									~(1,1)

After reduction of the domain for survey, the minimal set of pairs $\begin{pmatrix}m & m+n\\ n & n-m\end{pmatrix}$ in the matrix form are listed below.

Result 1: *Minimal pairs of 2D vectors forming $\pi/4$ in the single-digit domain. The first and second columns indicate $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively.*

$$\begin{pmatrix}0&1\\1&1\end{pmatrix},\begin{pmatrix}1&3\\2&1\end{pmatrix},\begin{pmatrix}1&5\\4&3\end{pmatrix},\begin{pmatrix}1&3\\5&2\end{pmatrix},\begin{pmatrix}1&7\\6&5\end{pmatrix},$$
$$\begin{pmatrix}1&4\\7&3\end{pmatrix},\begin{pmatrix}1&9\\8&7\end{pmatrix},\begin{pmatrix}1&5\\9&4\end{pmatrix},\begin{pmatrix}2&7\\5&3\end{pmatrix},\begin{pmatrix}2&9\\7&5\end{pmatrix}.$$

These 10 pairs of 2D vectors are referred to as "primary minimal pairs," representing the simplest and most fundamental configurations that generate a special angle within the single-digit domain. Other than the special angles defined here, there is the other easy-to-handle angle, $3\pi/4$, which is supplementary to $\pi/4$. Since this is an obtuse angle lying outside D_2, it cannot be found through the current approach. However, it can be obtained by modifying the data from **Result 1**. For example, $\boldsymbol{u} = (2,7)$ and $-\boldsymbol{v} = (-9,-5)$ form $3\pi/4$, because $\boldsymbol{u}$ and $\boldsymbol{v}$ form $\pi/4$.

A byproduct of the Proposition

Proposition derives an interesting consequence:

Corollary: *No grid points in a 2D space form an angle of $\pi/12$, $5\pi/12$, $\pi/8$, $3\pi/8$, $\pi/5$, $2\pi/5$.*

The proof makes for good exercise for advanced students, as it follows a similar approach to that in **Proposition**, given the following identities as a starting point:

$$\tan\frac{\pi}{12} = 2-\sqrt{3}, \qquad \tan\frac{5\pi}{12} = 2+\sqrt{3},$$
$$\tan\frac{\pi}{8} = \sqrt{2}-1, \qquad \tan\frac{3\pi}{8} = \sqrt{2}+1,$$
$$\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \qquad \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}.$$

In fact, proving the irrationality of $\sqrt{3}$ and $\sqrt{5}$ as well as $\sqrt{2}$, is not particularly difficult.

Preparation for the Numerical Survey in D_3

Suppose $\overrightarrow{\mathrm{AB}} = \boldsymbol{u} = (u_1, u_2, u_3)$ and $\overrightarrow{\mathrm{AC}} = \boldsymbol{v} = (v_1, v_2, v_3)$. Assume that $\boldsymbol{u}$ belongs to D_3 and seek $\boldsymbol{v}$ within D_3. Let $\gamma = \cos\angle\mathrm{BAC}$, then the relation

$$(\boldsymbol{u}\cdot\boldsymbol{v})^2 = \gamma^2\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2, \qquad (7)$$

where $\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$, $\|\boldsymbol{u}\|^2 = u_1^2 + u_2^2 + u_3^2$, *and* $\|\boldsymbol{v}\|^2 = v_1^2 + v_2^2 + v_3^2$. Thus, this leads to the following problem:

Q2: *Find $\boldsymbol{u}$ and $\boldsymbol{v}$ in D_3 satisfying the condition (7) with $\gamma^2 = 1/4, 1/2,$ or $3/4$.*

Condition (7) is a fourth order algebraic equation with six integer unknowns, placing it within the realm of number theory. A general, highly complex solution is unnecessary, as it would not be suitable for educational purposes. However, it is essential to obtain a fundamental set of vectors in D_3 that satisfy equation (7). To achieve this, the author concludes that a numerical survey of grid points is the most appropriate approach. Although the number of points in D_3 is large, it remains finite and is manageable using Python.

Before conducting the survey, the initial conditions for $\boldsymbol{u}$ are reduced according to properties (1) to (4). For the 3D case, the author planned a numerical survey. First, an AI chatbot was used to search for D_3 for all valid pairs $(\boldsymbol{u}, \boldsymbol{v})$, but it failed to find all the solutions. Next, the chatbot was used to assist in writing a Python script.

Primary Minimal Pairs in D_3

The Python script, refined with the assistance of an AI chatbot, successfully identified all possible grid points within D_3. However, it was less effective in reducing the output using properties (1) to (4). Therefore, the author manually reviewed and refined the results. The outcome is as follows:

Result 2: *11 pairs of 3D vectors $(\boldsymbol{u}, \boldsymbol{v})$ forming an angle of $\pi/6$ within D_3.*

$$\begin{pmatrix}0&1\\1&1\\1&2\end{pmatrix},\begin{pmatrix}0&1\\1&2\\1&7\end{pmatrix},\begin{pmatrix}0&1\\1&1\\7&2\end{pmatrix},\begin{pmatrix}1&3\\1&8\\2&5\end{pmatrix},$$
$$\begin{pmatrix}1&4\\1&5\\2&3\end{pmatrix},\begin{pmatrix}1&2\\1&5\\4&5\end{pmatrix},\begin{pmatrix}1&4\\2&1\\3&5\end{pmatrix},\begin{pmatrix}1&4\\2&3\\7&5\end{pmatrix},$$
$$\begin{pmatrix}1&5\\3&2\\4&7\end{pmatrix},\begin{pmatrix}1&3\\7&2\\8&5\end{pmatrix},\begin{pmatrix}2&5\\5&3\\5&4\end{pmatrix}.$$

Result 3: *13 pairs of 3D vectors $(\boldsymbol{u}, \boldsymbol{v})$ forming an angle of $\pi/4$ within D_3.*

$$\begin{pmatrix}0&3\\0&4\\1&5\end{pmatrix},\begin{pmatrix}0&2\\1&1\\1&2\end{pmatrix},\begin{pmatrix}0&4\\1&1\\1&8\end{pmatrix},\begin{pmatrix}0&6\\1&1\\1&9\end{pmatrix},\begin{pmatrix}0&5\\1&7\\2&4\end{pmatrix},$$
$$\begin{pmatrix}0&2\\1&1\\7&2\end{pmatrix},\begin{pmatrix}0&6\\1&6\\7&7\end{pmatrix},\begin{pmatrix}0&5\\3&3\\4&4\end{pmatrix},\begin{pmatrix}1&4\\1&7\\4&4\end{pmatrix},$$
$$\begin{pmatrix}1&7\\3&8\\4&2\end{pmatrix},\begin{pmatrix}1&5\\4&4\\8&3\end{pmatrix},\begin{pmatrix}1&4\\5&5\\8&2\end{pmatrix},\begin{pmatrix}2&8\\3&5\\6&3\end{pmatrix}.$$

Result 4: *4 pairs of 3D vectors $(\boldsymbol{u}, \boldsymbol{v})$ forming an angle of $\pi/3$ within D_3.*

$$\begin{pmatrix}0&1\\1&0\\1&1\end{pmatrix},\begin{pmatrix}0&5\\1&4\\7&3\end{pmatrix},\begin{pmatrix}1&1\\1&4\\4&1\end{pmatrix},\begin{pmatrix}1&4\\4&9\\9&1\end{pmatrix}.$$

These 28 pairs of 3D vectors are also referred to as "primary minimal pairs."

Complementary Integer Adjustment

Although the primary minimal pairs seem sufficient for educational purposes, the author introduces two theoretical approaches to systematically generate additional vector pairs and enrich the set: "complementary integer adjustment" (CIA) and "rational orthogonal transformation" (ROT). As a result, the outcomes obtained via CIA and ROT include some pairs lying outside the domain D_3.

When given a special angle $\theta = A(\boldsymbol{u}, \boldsymbol{v})$, its complementary angle $\pi/2 - \theta$ can be found by rotating $\boldsymbol{v}$ around $\boldsymbol{u}$ by $\pi/2$. However, this may result in an irrational expression. Instead, we assume $\boldsymbol{w} = a\boldsymbol{u} + b\boldsymbol{v}$ such that $\boldsymbol{w} \perp \boldsymbol{v}$, ensuring

$$A(\boldsymbol{u}, \boldsymbol{w}) = \frac{\pi}{2} - A(\boldsymbol{u}, \boldsymbol{v}). \tag{8}$$

A dot product of $\boldsymbol{w}$ and $\boldsymbol{v}$ leads to an indeterminate equation for a and b: $a\boldsymbol{u} \cdot \boldsymbol{v} + b\boldsymbol{v} \cdot \boldsymbol{v} = 0$, satisfied with $a : b = \|\boldsymbol{v}\|^2 : (-\boldsymbol{u} \cdot \boldsymbol{v})$. Thus, we define

$$\boldsymbol{w} \stackrel{\text{def}}{=} \|\boldsymbol{v}\|^2 \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v}. \tag{9}$$

The vector $\boldsymbol{w}$ given by (9) satisfies (8). Indeed, since

$$\boldsymbol{w} \cdot \boldsymbol{u} = \|\boldsymbol{v}\|^2 \|\boldsymbol{u}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 \sin^2 \theta$$

and

$$\boldsymbol{w}^2 = (\|\boldsymbol{v}\|^2 \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{v})^2 = \|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^4 \sin^2 \theta,$$

we obtain

$$\cos\{A(\boldsymbol{u}, \boldsymbol{w})\} = \sin \theta = \cos\left(\frac{\pi}{2} - \theta\right),$$

which confirms (8).

Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are integer vector, $\boldsymbol{w}$ is also an integer vector. To ensure that the coordinates of $\boldsymbol{w}$ be mutually prime, we divide $\boldsymbol{w}$ by the greatest common divisor (GCD) of its coordinates:

$$\boldsymbol{w}' = \frac{1}{\omega} \boldsymbol{w}, \tag{10}$$

where ω is GCD of the coordinates of $\boldsymbol{w}$. This process is referred to as a "complementary integer adjustment" (CIA). Thus, we say that $\boldsymbol{w}'$ is the CIA of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$. Similarly, we determine the CIA of $\boldsymbol{u}$ with respect to $\boldsymbol{v}$ by $\widetilde{\boldsymbol{w}} \stackrel{\text{def}}{=} \|\boldsymbol{u}\|^2 \boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{u}$; therefore,

$$\boldsymbol{w}" = \frac{1}{\omega'} \widetilde{\boldsymbol{w}} \tag{11}$$

where ω' is GCD of the coordinates of $\widetilde{\boldsymbol{w}}$.

Results by CIA outside D_3

The CIA process was applied to $\boldsymbol{v}$ (respectively, $\boldsymbol{u}$) of **Result 2** with respect to $\boldsymbol{u}$ (respectively, $\boldsymbol{v}$). The outcomes are presented as follows.

Result 5: *Pairs of 3D vectors forming an angle of* $\pi/3$ *and lying outside* D_3, *listed as 7 pairs of* $(\boldsymbol{u}, \boldsymbol{w}')$ *followed by 8 pairs of* $(\boldsymbol{v}, \boldsymbol{w}")$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 4 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -7 \\ 7 & 8 \\ 8 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 4 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & -2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 3 & 4 \\ 4 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ 1 & -3 \\ 7 & 4 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 7 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 5 & 5 \\ 3 & -4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 5 & 7 \\ 5 & -2 \end{pmatrix},$$
$$\begin{pmatrix} 4 & 5 \\ 1 & -4 \\ 5 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 2 & -3 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 8 & 9 \\ 5 & -4 \end{pmatrix}, \begin{pmatrix} 5 & 7 \\ 2 & -5 \\ 7 & 2 \end{pmatrix}.$$

The CIA process was applied to $\boldsymbol{v}$ (respectively, $\boldsymbol{u}$) of **Result 3** with respect to $\boldsymbol{u}$ (respectively, $\boldsymbol{v}$). The outcomes are presented as follows.

Result 6: *Pairs of 3D vectors forming an angle of* $\pi/4$ *and lying outside* D_3, *compiled into 9 pairs of* $(\boldsymbol{u}, \boldsymbol{w}')$ *and 5 pairs of* $(\boldsymbol{v}, \boldsymbol{w}")$, *with 2 pairs omitted because they already appear in* **Result 3**.

$$\begin{pmatrix} 0 & -3 \\ 0 & -4 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -4 \\ 1 & 8 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -6 \\ 1 & 9 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ 1 & -1 \\ 2 & 8 \end{pmatrix},$$
$$\begin{pmatrix} 0 & -5 \\ 3 & 3 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & 0 \\ 8 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -4 \\ 3 & 1 \\ 6 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -4 \\ 4 & 8 \end{pmatrix};$$
$$\begin{pmatrix} 2 & 4 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 8 \\ 1 & -7 \\ 8 & 7 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 7 & 4 \\ 4 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 7 \\ 5 & 5 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} 8 & 6 \\ 5 & 2 \\ 3 & -3 \end{pmatrix}$$

The CIA process was applied to $\boldsymbol{v}$ (respectively, $\boldsymbol{u}$) of **Result 4** with respect to $\boldsymbol{u}$ (respectively, $\boldsymbol{v}$). The outcomes are presented as follows.

Result 7: *Pairs of 3D vectors forming an angle of* $\pi/6$ *and lying outside* D_3, *listed as 2 pairs for* $(\boldsymbol{u}, \boldsymbol{w}')$ *followed by 3 pairs for* $(\boldsymbol{v}, \boldsymbol{w}")$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -2 \\ 7 & 7 \end{pmatrix};$$
$$\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 4 & 7 \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 9 & 2 \\ 1 & -1 \end{pmatrix}.$$

Rational Orthogonal Transformation

When the primary minimal pairs are transformed by orthogonal matrices, their image also form special angles. If all entries of the orthogonal matrices are rational, then - after suitable scalar multiplications - the transformed vectors may still lie in the single-digit domain.

Suppose (p, q, r, s) to form a Pythagorean quadruples, *i.e.*, $p^2 + q^2 + r^2 = s^2$, which implies

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \tag{12}$$

where $\alpha = \pm p/s$, $\beta = \pm q/s$ and $\gamma = \pm r/s$. Now define row vectors

$$\boldsymbol{u} = [\alpha, \beta, \gamma], \boldsymbol{v} = [\beta, \gamma, \alpha], \boldsymbol{w} = [\gamma, \alpha, \beta] \tag{13}$$

By adjusting signs appropriately to ensure

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{u} = 0 \tag{14}$$

these vectors can be stacked (in any order) to form a rational orthogonal matrix of size 3×3. In fact, a wide variety of rational orthogonal matrices can be constructed using this approach. For this article, we propose the following parameterized form by a positive integer n of Pythagorean quadruples:

$$(p, q, r, s) = (n, n + 1, n(n + 1), n^2 + n + 1).$$

While existing literature (*e.g.*, Liebeck and Osborne, 1991) provides general constructions using Cayley transforms, our approach focuses on explicit and elementary constructions using Pythagorean quadruples, offering a more accessible path for both computational purposes and educational applications. In this parameterization, the quadruples for $n = 1$ and 2 are $(1,2,2,3)$ and $(2,3,6,7)$, respectively. For $n \geq 3$, some entries exceed the single-digit domain and are thus omitted. Examples of rational orthogonal matrices constructed in this way are:

$$O_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, O_2 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix},$$

$$O_3 = \begin{pmatrix} \frac{6}{7} & \frac{2}{7} & \frac{3}{7} \\ -\frac{3}{7} & \frac{6}{7} & \frac{2}{7} \\ -\frac{2}{7} & -\frac{3}{7} & \frac{6}{7} \end{pmatrix}.$$

The matrices O_1, O_2, and O_3 represent rotations around the vectors $[0,2,-1]^T$, $[3,1,1]^T$, and $[1,-1,1]^T$. Let ROT denote the "rational orthogonal transformation."

Results by ROT outside D_3

Applying ROT using O_1, O_2 and O_3 to **Result 2** yields outcomes that exceed D_3. Therefore, we select pairs that contain negative but single-digit entries. Moreover, the outcomes obtained via O_2 overlap with those via O_1 and are thus omitted here. Most outcomes via O_3 result in two-digit entries, leaving only two usable pairs. Only two pairs remain.

Result 8: *Pairs of 3D vectors* $(\boldsymbol{u}, \boldsymbol{v})$ *via* O_1, *include: 7 pairs with* $A(\boldsymbol{u}, \boldsymbol{v}) = \pi/6$, *4 pairs with* $A(\boldsymbol{u}, \boldsymbol{v}) = \pi/4$, *and 1 pair with* $A(\boldsymbol{u}, \boldsymbol{v}) = \pi/3$. *Pairs via* O_3 *are reduced to 1 pair with* $A(\boldsymbol{u}, \boldsymbol{v}) = \pi/6$ *and 1 pair with* $A(\boldsymbol{u}, \boldsymbol{v}) = \pi/3$.

$$\begin{pmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -5 & -1 \\ 4 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} -1 & -4 \\ 1 & -1 \\ 2 & 9 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 2 & 7 \end{pmatrix},$$

$$\begin{pmatrix} -2 & -1 \\ 1 & 4 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} -3 & -2 \\ 1 & 5 \\ 4 & 7 \end{pmatrix}, \begin{pmatrix} -7 & -2 \\ 1 & 3 \\ 8 & 5 \end{pmatrix};$$

$$\begin{pmatrix} -1 & -1 \\ 0 & 4 \\ 1 & 8 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} -5 & -1 \\ 4 & 4 \\ 3 & 8 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 1 & 1 \\ 2 & 7 \end{pmatrix};$$

$$\begin{pmatrix} -5 & 0 \\ 4 & 1 \\ 3 & 7 \end{pmatrix}; \begin{pmatrix} -5 & -2 \\ 3 & -1 \\ 8 & 7 \end{pmatrix}; \begin{pmatrix} -5 & 4 \\ 3 & -1 \\ 8 & 9 \end{pmatrix}.$$

Summary

In this article, we explored grid points in the single-digit domain of D_2 and D_3 that generate special angles: $\pi/6$, $\pi/4$, and $\pi/3$. We theoretically identified 10 pairs of 2D vectors and numerically found 28 pairs of 3D vectors. Two theoretical tools, CIA and ROT, were introduced to complement and extend the primary minimal pairs. Through this process, additional 48 pairs of 3D integer vectors forming the special angles were identified just outside D_3.

Educators may use these samples as a basis for designing original exercises that deepen students' understanding of integer vector geometry and angle formation. As a byproduct of the theoretical approach to the 2D case, we found that no grid points can generate angles of $\pi/6$ or $\pi/3$. This result can naturally be extended to other angles, which in turn leads to another valuable exercise for advanced students: proving that no 2D grid points generate $\pi/12$, $5\pi/12$, $\pi/8$, $3\pi/8$, $\pi/5$, $2\pi/5$, and their supplementary angles.

Furthermore, the systematic approach developed in this study could be extended to higher dimensions or different number domains, opening new avenues for research and pedagogical applications. Future work may explore whether similar constraints exist in 4D integer spaces or extend the analysis to alternative domains such as Gaussian or Eisenstein integers.

Limitations and Future work

Before formally implementing this approach in the classroom, we received informal feedback from colleagues suggesting that it could help promote deeper student engagement with basic formulas.

While this study has successfully identified minimal pairs of integer vectors generating special angles within the single-digit domain, several limitations remain.

First, although the numerical survey appears comprehensive, no formal proof has yet been provided.

Second, the author employed three matrices, O_1, O_2, and O_3, as instances of ROT; however, they were not sufficient. Many more rational orthogonal matrices exist.

Third, the analysis focused exclusively on 2D and 3D integer spaces. Extending the methodology to higher dimensions (*e.g.*, 4D) or to alternative number domains, such as Gaussian or Eisenstein integers, remains an open and promising direction for future research.

These avenues offer opportunities to deepen both mathematical theory and its educational applications.

Although this paper does not provide empirical data on student learning, a classroom trial using the identified vector pairs is planned for the next academic semester. The author intends to analyze student feedback and performance in exercises designed around these minimal pairs.

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